A change of variable formula for the 2D fractional Brownian motion of Hurst index bigger or equal to 1/4

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A change of variable formula for the 2D fractional Brownian motion of Hurst index bigger or equal to $1/4$

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Abstract: We prove a change of variable formula for the 2D fractional Brownian motion of index H bigger or equal to $1/4$. For H strictly bigger than $1/4$, our formula coincides with that obtained by using the rough paths theory. For $H = 1/4$ (the more interesting case), there is an additional term that is a classical Wiener integral against an independent standard Brownian motion.

Key words: Fractional Brownian motion; weak convergence; change of variable formula.

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1 Introduction and main result

In [4], Coutin and Qian have shown that the rough paths theory of Lyons [13] can be applied to the 2D fractional Brownian motion $B = (B^{(1)}, B^{(2)})$ under the condition that its Hurst parameter H (supposed to be the same for the two components) is *strictly* bigger than $1/4$. Since this seminal work, several authors have recovered this fact by using different routes (see e.g. Feyel and de La Pradelle [7], Friz and Victoir [8] or Unterberger [19] to cite but a few). On the other hand, it is still an open problem to bypass this restriction on H.

Rough paths theory is purely deterministic in essence. Actually, its random aspect comes only when it is applied to a single path of a given *stochastic* process (like a Brownian motion, a fractional Brownian motion, etc.). In particular, *it does not allow to produce a new alea*. As such, the second point of Theorem 1.2 just below shows, in a sense, that it seems difficult to reach the case $H = 1/4$ by using exclusively the tools of rough paths theory.

Before stating our main result, we need some preliminaries. Let W be a standard (1D) Brownian motion, independent of B. We assume that B and W are defined on the same probability space $(\Omega, \mathscr{F}, P)$ with $\mathscr{F} = \sigma\{B\} \vee \sigma\{W\}$. Let (X_n) be a sequence of $\sigma\{B\}$-measurable random variables, and let X be a $\mathscr{F}$-measurable random variable. In the sequel, we will write $X_n \overset{\text{stably}}{\longrightarrow} X$ if $(Z, X_n) \overset{\text{law}}{\longrightarrow} (Z, X)$ for all bounded and $\sigma\{B\}$-measurable random variable Z. In particular, we see that the stable convergence imply the convergence in law. Moreover, it is easily checked that the convergence in probability implies the stable convergence. We refer to [11] for an exhaustive study of this notion.

Now, let us introduce the following object:

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Definition 1.1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, and fix a time $t > 0$. Provided it exists, we define $\int_0^t \nabla f(B_s) \cdot dB_s$ to be the limit in probability, as $n \to \infty$, of*

$$\begin{aligned} I_n(t) \;&:=\; \sum_{k=0}^{\lfloor nt \rfloor -1} \frac{\frac{\partial f}{\partial x}(B^{(1)}_{k/n}, B^{(2)}_{k/n}) + \frac{\partial f}{\partial x}(B^{(1)}_{(k+1)/n}, B^{(2)}_{k/n})}{2} \big(B^{(1)}_{(k+1)/n} - B^{(1)}_{k/n}\big) \\ &+ \sum_{k=0}^{\lfloor nt \rfloor -1} \frac{\frac{\partial f}{\partial y}(B^{(1)}_{k/n}, B^{(2)}_{k/n}) + \frac{\partial f}{\partial y}(B^{(1)}_{k/n}, B^{(2)}_{(k+1)/n})}{2} \big(B^{(2)}_{(k+1)/n} - B^{(2)}_{k/n}\big). \end{aligned} \tag{1.1}$$

If $I_n(t)$ defined by (1.1) does not converge in probability but converges stably, we denote the limit by $\int_0^t \nabla f(B_s) \cdot d^\star B_s$.

Our main result is as follows:

Theorem 1.2 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function belonging to $\mathscr{C}^8$ and verifying $(\mathbf{H_8})$, see (3.15) below. Let also $B = (B^{(1)}, B^{(2)})$ denote a 2D fractional Brownian motion of Hurst index $H \in (0,1)$, and $t > 0$ be a fixed time.*

1. *If $H > 1/4$ then $\int_0^t \nabla f(B_s) \cdot dB_s$ is well-defined, and we have*

$$f(B_t) = f(0) + \int_0^t \nabla f(B_s) \cdot dB_s. \tag{1.2}$$

2. *If $H = 1/4$ then only $\int_0^t \nabla f(B_s) \cdot d^\star B_s$ is well-defined, and we have*

$$f(B_t) \overset{\text{Law}}{=} f(0) + \int_0^t \nabla f(B_s) \cdot d^\star B_s + \frac{\sigma_{1/4}}{\sqrt{2}} \int_0^t \frac{\partial^2 f}{\partial x \partial y}(B_s) dW_s. \tag{1.3}$$

   Here, $\sigma_{1/4}$ is the universal constant defined below by (1.5), and $\int_0^t \partial^2 f/\partial x\partial y(B_s)dW_s$ denotes a classical Wiener integral with respect to the independent Brownian motion W.

3. *If $H < 1/4$ then the integral $\int_0^t B_s \cdot d^\star B_s$ does not exist. Therefore, it is not possible to write a change of variable formula for $B_t^{(1)} B_t^{(2)}$ using the integral defined in Definition 1.1.*

Remark 1.3

1. Due to the definition of the stable convergence, we can freely move each component in (1.3) from the right hand side to the left (or from the left hand side to the right).

2. Whenever β denotes a one-dimensional fractional Brownian motion with Hurst index in $(0, 1/2)$, it is easily checked, for any fixed $t > 0$, that $\sum_{k=0}^{\lfloor nt \rfloor -1} \beta_{k/n}\big(\beta_{(k+1)/n} - \beta_{k/n}\big)$ does not converge in law. $\big($Indeed, on one hand, we have

$$\beta^2_{\lfloor nt \rfloor/t} = \sum_{k=0}^{\lfloor nt \rfloor -1} \big(\beta^2_{(k+1)/n} - \beta^2_{k/n}\big) = 2 \sum_{k=0}^{\lfloor nt \rfloor -1} \beta_{k/n}\big(\beta_{(k+1)/n} - \beta_{k/n}\big) + \sum_{k=0}^{\lfloor nt \rfloor -1} \big(\beta_{(k+1)/n} - \beta_{k/n}\big)^2$$

   and, on the other hand, it is well-known (see e.g. [12]) that

$$n^{2H-1} \sum_{k=0}^{\lfloor nt \rfloor -1} \big(\beta_{(k+1)/n} - \beta_{k/n}\big)^2 \xrightarrow[n \to \infty]{L^2} t.$$

These two facts imply immediately that

$$\sum_{k=0}^{\lfloor nt\rfloor-1} \beta_{k/n}\big(\beta_{(k+1)/n}-\beta_{k/n}\big) = \frac{1}{2}\left(\beta^2_{\lfloor nt\rfloor/t} - \sum_{k=0}^{\lfloor nt\rfloor-1} \big(\beta_{(k+1)/n}-\beta_{k/n}\big)^2\right)$$

does not converge in law$\big)$. On the other hand, whenever $H>1/6$, the quantity

$$\sum_{k=0}^{\lfloor nt\rfloor-1} \frac{1}{2}\big(f(\beta_{k/n})+f(\beta_{(k+1)/n})\big)\left(\beta_{(k+1)/n}-\beta_{k/n}\right)$$

converges in L^2 for any regular enough function $f:\mathbb{R}\to\mathbb{R}$, see [9] and [3]. This last fact roughly explains why there is a "symmetric" part in the Riemann sum (1.1).

3. We stress that it is still an open problem to know if each individual integral $\int_0^t \frac{\partial f}{\partial x}(B_s)d^{(\star)}B_s^{(1)}$ and $\int_0^t \frac{\partial f}{\partial y}(B_s)d^{(\star)}B_s^{(2)}$ could be defined separately. Indeed, in the first two points of Theorem 1.2, we "only" prove that their sum, that is $\int_0^t \nabla f(B_s)\cdot d^{(\star)}B_s$, is well-defined.

4. Let us give a quicker proof of (1.3) in the particular case where $f(x,y)=xy$. Let β be a one-dimensional fractional Brownian motion of index 1/4. The classical Breuer-Major's theorem [1] yields:

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor n\cdot\rfloor-1}\big(\sqrt{n}(\beta_{(k+1)/n}-\beta_{k/n})^2-1\big) \overset{\text{Law}}{=} \frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor n\cdot\rfloor-1}\big((\beta_{k+1}-\beta_k)^2-1\big) \underset{n\to\infty}{\overset{\text{stably}}{\longrightarrow}} \sigma_{1/4}\,W. \quad (1.4)$$

Here, the convergence is stable and holds in the Skorohod space $\mathscr{D}$ of càdlàg functions on $[0,\infty)$. Moreover, W still denotes a standard Brownian motion independent of β (the independence is a consequence of the central limit theorem for multiple stochastic integrals proved in [18]) and the constant $\sigma_{1/4}$ is given by

$$\sigma_{1/4} := \sqrt{\frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\sqrt{|k+1|}+\sqrt{|k-1|}-2\sqrt{|k|}\right)^2} < \infty. \quad (1.5)$$

Now, let $\widetilde{\beta}$ be another fractional Brownian motion of index 1/4, independent of β. From (1.4), we get

$$\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor-1}\big(\sqrt{n}(\beta_{(k+1)/n}-\beta_{k/n})^2-1\big), \frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor-1}\big(\sqrt{n}(\widetilde{\beta}_{(k+1)/n}-\widetilde{\beta}_{k/n})^2-1\big)\right) \underset{n\to\infty}{\overset{\text{stably}}{\longrightarrow}} \sigma_{1/4}\,(W,\widetilde{W})$$

for $(W,\widetilde{W})$ a 2D standard Brownian motion, independent of the 2D fractional Brownian motion $(\beta,\widetilde{\beta})$. In particular, by difference, we have

$$\frac{1}{2}\sum_{k=0}^{\lfloor n\cdot\rfloor-1}\big((\beta_{(k+1)/n}-\beta_{k/n})^2-(\widetilde{\beta}_{(k+1)/n}-\widetilde{\beta}_{k/n})^2\big) \underset{n\to\infty}{\overset{\text{stably}}{\longrightarrow}} \frac{\sigma_{1/4}}{2}(W-\widetilde{W}) \overset{\text{Law}}{=} \frac{\sigma_{1/4}}{\sqrt{2}}\,W.$$

Now, set $B^{(1)} = (\beta + \widetilde{\beta})/\sqrt{2}$ and $B^{(2)} = (\beta - \widetilde{\beta})/\sqrt{2}$. It is easily checked that $B^{(1)}$ and $B^{(2)}$ are two independent fractional Brownian motions of index 1/4. Moreover, we can rewrite the previous convergence as

$$\sum_{k=0}^{\lfloor n\cdot\rfloor -1} (B^{(1)}_{(k+1)/n} - B^{(1)}_{k/n})(B^{(2)}_{(k+1)/n} - B^{(2)}_{k/n}) \xrightarrow[n\to\infty]{\text{stably}} \frac{\sigma_{1/4}}{\sqrt{2}} W, \tag{1.6}$$

with $B^{(1)}$, $B^{(2)}$ and W independent. On the other hand, for any $a, b, c, d \in \mathbb{R}$:

$$bd - ac = a(d-c) + c(b-a) + (b-a)(d-c).$$

Choosing $a = B^{(1)}_{k/n}$, $b = B^{(1)}_{(k+1)/n}$, $c = B^{(2)}_{k/n}$ and $d = B^{(2)}_{(k+1)/n}$, and suming for k over $0, \ldots, \lfloor nt\rfloor - 1$, we obtain

$$\begin{aligned} B^{(1)}_{\lfloor nt\rfloor/n} B^{(2)}_{\lfloor nt\rfloor/n} &= \sum_{k=0}^{\lfloor nt\rfloor -1} B^{(1)}_{k/n}\big(B^{(2)}_{(k+1)/n} - B^{(2)}_{k/n}\big) + B^{(2)}_{k/n}\big(B^{(1)}_{(k+1)/n} - B^{(1)}_{k/n}\big) \\ &\quad + \sum_{k=0}^{\lfloor nt\rfloor -1} \big(B^{(1)}_{(k+1)/n} - B^{(1)}_{k/n}\big)\big(B^{(2)}_{(k+1)/n} - B^{(2)}_{k/n}\big). \end{aligned} \tag{1.7}$$

Hence, passing to the limit using (1.6), we get the desired conclusion in (1.3), in the particular case where $f(x, y) = xy$. Note that the second term in the right-hand side of (1.7) is the discrete analogue of the 2-*covariation* introduced by Errami and Russo in [6].

5. We could prove (1.3) at a functional level (note that it has precisely been done for $f(x, y) = xy$ in the proof just below). But, in order to keep the length of this paper within limits, we defer to future analysis this rather technical investigation.

6. In the very recent work [16], Réveillac and I proved the following result (see also Burdzy and Swanson [2] for similar results in the case where β is replaced by the solution of the stochastic heat equation driven by a space/time white noise). If β denotes a one-dimensional fractional Brownian motion of index 1/4 and if $g : \mathbb{R} \to \mathbb{R}$ is regular enough, then

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g(\beta_{k/n})\big(\sqrt{n}(\beta_{(k+1)/n} - \beta_{k/n})^2 - 1\big) \xrightarrow[n\to\infty]{\text{stably}} \frac{1}{4}\int_0^1 g''(\beta_s)ds + \sigma_{1/4}\int_0^1 g(\beta_s)dW_s \tag{1.8}$$

for W a standard Brownian motion independent of β. Compare with Proposition 3.3 below. In particular, by choosing g identically one in (1.8), it agrees with (1.4).

7. The fractional Brownian motion of index 1/4 has a remarkable physical interpretation in terms of particle systems. Indeed, if one consider an infinite number of particles, initially placed on the real line according to a Poisson distribution, performing independent Brownian motions and undergoing "elastic" collisions, then the trajectory of a fixed particle (after rescaling) converges to a fractional Brownian motion of index 1/4. See Harris [10] for heuristic arguments, and Dürr, Goldstein and Lebowitz [5] for precise results.

Now, the rest of the note is entirely devoted to the proof of Theorem 1.2. The Section 2 contains some preliminaries and fix the notation. Some technical results are postponed in Section 3. Finally, the proof of Theorem 1.2 is done in Section 4.

2 Preliminaries and notation

We shall now provide a short description of the tools of Malliavin calculus that will be needed in the following sections. The reader is referred to the monographs [14] and [17] for any unexplained notion or result.

Let $B = (B_t^{(1)}, B_t^{(2)})_{t\in[0,T]}$ be a 2D fractional Brownian motion with Hurst parameter belonging to $(0, 1/2)$. We denote by $\mathcal{H}$ the Hilbert space defined as the closure of the set of step $\mathbb{R}^2$-valued functions on $[0, T]$, with respect to the scalar product induced by

$$\big\langle \big(\mathbf{1}_{[0,t_1]}, \mathbf{1}_{[0,t_2]}\big), \big(\mathbf{1}_{[0,s_1]}, \mathbf{1}_{[0,s_2]}\big)\big\rangle_{\mathcal{H}} = R_H(t_1, s_1) + R_H(t_2, s_2), \quad s_i, t_i \in [0, T], \quad i = 1, 2,$$

where $R_H(t, s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$. The mapping $(\mathbf{1}_{[0,t_1]}, \mathbf{1}_{[0,t_2]}) \mapsto B_{t_1}^{(1)} + B_{t_2}^{(2)}$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with B. Also, $\mathfrak{H}$ will denote the Hilbert space defined as the closure of the set of step $\mathbb{R}$-valued functions on $[0, T]$, with respect to the scalar product induced by

$$\big\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\big\rangle_{\mathfrak{H}} = R_H(t, s), \quad s, t \in [0, T].$$

The mapping $\mathbf{1}_{[0,t]} \mapsto B_t^{(i)}$ (i equals 1 or 2) can be extended to an isometry between $\mathfrak{H}$ and the Gaussian space associated with $B^{(i)}$.

Consider the set of all smooth cylindrical random variables, i.e. of the form

$$F = f\big(B(\varphi_1), \ldots, B(\varphi_k)\big), \quad \varphi_i \in \mathcal{H}, \quad i = 1, \ldots, k, \tag{2.9}$$

where $f \in \mathscr{C}^\infty$ is bounded with bounded derivatives. The derivative operator D of a smooth cylindrical random variable of the above form is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}\big(B(\varphi_1), \ldots, B(\varphi_k)\big)\varphi_i =: \big(D_{B^{(1)}}F, D_{B^{(2)}}F\big).$$

In particular, we have

$$D_{B^{(i)}} B_t^{(j)} = \delta_{ij}\mathbf{1}_{[0,t]} \quad \text{for } i, j \in \{1, 2\}, \text{ and } \delta_{ij} \text{ the Kronecker symbol.}$$

By iteration, one can define the mth derivative $D^m F$ (which is a symmetric element of $L^2(\Omega, \mathcal{H}^{\otimes m})$) for $m \geqslant 2$. As usual, for any $m \geqslant 1$, the space $\mathbb{D}^{m,2}$ denotes the closure of the set of smooth random variables with respect to the norm $\|\cdot\|_{m,2}$ defined by the relation

$$\|F\|_{m,2}^2 = E|F|^2 + \sum_{i=1}^{m} E\|D^i F\|_{\mathcal{H}^{\otimes i}}^2.$$

The derivative D verifies the chain rule. Precisely, if $\varphi : \mathbb{R}^n \to \mathbb{R}$ belongs to $\mathscr{C}^1$ with bounded derivatives and if F_i, $i = 1, \ldots, n$, are in $\mathbb{D}^{1,2}$, then $\varphi(F_1, \ldots, F_n) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F_1, \ldots, F_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, \ldots, F_n) DF_i.$$

The mth derivative $D^m_{B^{(i)}}$ (i equals 1 or 2) verifies the following Leibnitz rule: for any $F, G \in \mathbb{D}^{m,2}$ such that $FG \in \mathbb{D}^{m,2}$, we have

$$\big(D^m_{B^{(i)}}FG\big)_{t_1,\ldots,t_m} = \sum \big(D^r_{B^{(i)}}F\big)_{s_1,\ldots,s_r}\big(D^{m-r}_{B^{(i)}}G\big)_{u_1,\ldots,u_{m-r}}, \quad t_i \in [0,T], \quad i = 1,\ldots,m, \tag{2.10}$$

where the sum runs over any subset $\{s_1,\ldots,s_r\} \subset \{t_1,\ldots,t_m\}$ and where we write $\{t_1,\ldots,t_m\} \setminus \{s_1,\ldots,s_r\} =: \{u_1,\ldots,u_{m-r}\}$.

The divergence operator δ is the adjoint of the derivative operator. If a random variable $u \in L^2(\Omega,\mathcal{H})$ belongs to domδ, the domain of the divergence operator, then $\delta(u)$ is defined by the duality relationship

$$E\big(F\delta(u)\big) = E\langle DF, u\rangle_{\mathcal{H}}$$

for every $F \in \mathbb{D}^{1,2}$.

For every $q \geqslant 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of B, that is, the closed linear subspace of $L^2(\Omega,\mathcal{A},P)$ generated by the random variables $\{H_q(B(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where H_q is the qth Hermite polynomial given by $H_q(x) = (-1)^q e^{x^2/2}\frac{d^q}{dx^q}\big(e^{-x^2/2}\big)$. The mapping

$$I_q(h^{\otimes q}) = H_q(B(h)) \tag{2.11}$$

provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot q}$ (equipped with the modified norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$) and $\mathcal{H}_q$. The following duality formula holds

$$E\left(FI_q(f)\right) = E\left(\langle D^q F, f\rangle_{\mathcal{H}^{\otimes q}}\right), \tag{2.12}$$

for any $f \in \mathcal{H}^{\odot q}$ and $F \in \mathbb{D}^{q,2}$. In particular, we have

$$E\left(FI^{(i)}_q(g)\right) = E\left(\left\langle D^q_{B^{(i)}}F, g\right\rangle_{\mathfrak{H}^{\otimes q}}\right), \quad i = 1,2, \tag{2.13}$$

for any $g \in \mathfrak{H}^{\odot q}$ and $F \in \mathbb{D}^{q,2}$, where, for simplicity, we write $I^{(i)}_q(g)$ whenever the corresponding qth multiple integral is only with respect to $B^{(i)}$.

Finally, we mention the following particular case (actually, the only one we will need in the sequel) of the classical multiplication formula: if $f, g \in \mathfrak{H}$, $q \geqslant 1$ and $i \in \{1,2\}$, then

$$I^{(i)}_q(f^{\otimes q})I^{(i)}_q(g^{\otimes q}) = \sum_{r=0}^{q} r!\binom{q}{r}^2 I^{(i)}_{2q-2r}(f^{\otimes q-r} \otimes g^{\otimes q-r})\langle f,g\rangle^r_{\mathfrak{H}}. \tag{2.14}$$

3 Some technical results

In this section, we collect some crucial results for the proof of (1.3), the only case which is difficult.

Here and in the rest of the paper, we set

$$\Delta B^{(i)}_{k/n} := B^{(i)}_{(k+1)/n} - B^{(i)}_{k/n}, \quad \delta_{k/n} := \mathbf{1}_{[k/n,(k+1)/n]} \quad \text{and} \quad \varepsilon_{k/n} := \mathbf{1}_{[0,k/n]},$$

for any $i \in \{1,2\}$ and $k \in \{0,\ldots,n-1\}$.

In the sequel, for $g : \mathbb{R}^2 \to \mathbb{R}$ belonging to $\mathscr{C}^q$, we will need assumption of the type:

$$(\mathbf{H_q}) \quad \sup_{s\in[0,1]} E\left|\frac{\partial^{a+b}g}{\partial x^a \partial y^b}(B^{(1)}_s, B^{(2)}_s)\right|^p < \infty \text{ for all } p \geqslant 1 \text{ and all integers } a,b \geqslant 0 \text{ s.t. } a+b \leqslant q.$$

(3.15)

We begin by the following technical lemma:

Lemma 3.1 *Let β be a 1D fractional Brownian motion of Hurst index* $1/4$. *We have*

(i) $\big|E\big(\beta_r(\beta_t-\beta_s)\big)\big| \leqslant \sqrt{|t-s|}$ *for any* $0\leqslant r,s,t\leqslant 1$,

(ii) $\displaystyle\sum_{k,l=0}^{n-1}\Big|\big\langle\varepsilon_{l/n},\delta_{k/n}\big\rangle_{\mathfrak{H}}\Big| \underset{n\to\infty}{=} O(n)$,

(iii) $\displaystyle\sum_{k,l=0}^{n-1}\Big|\big\langle\delta_{l/n},\delta_{k/n}\big\rangle_{\mathfrak{H}}\Big|^r \underset{n\to\infty}{=} O(n^{1-r/2})$ *for any* $r\geqslant 1$,

(iv) $\displaystyle\sum_{k=0}^{n-1}\left|\big\langle\varepsilon_{k/n},\delta_{k/n}\big\rangle_{\mathfrak{H}}+\frac{1}{2\sqrt{n}}\right| \underset{n\to\infty}{=} O(1)$,

(v) $\displaystyle\sum_{k=0}^{n-1}\left|\big\langle\varepsilon_{k/n},\delta_{k/n}\big\rangle_{\mathfrak{H}}^2-\frac{1}{4n}\right| \underset{n\to\infty}{=} O(1/\sqrt{n})$.

Proof of Lemma 3.1.

(i) We have
$$E\big(\beta_r(\beta_t-\beta_s)\big)=\frac{1}{2}\big(\sqrt{t}-\sqrt{s}\big)+\frac{1}{2}\left(\sqrt{|s-r|}-\sqrt{|t-r|}\right).$$
Using the classical inequality $\left|\sqrt{|b|}-\sqrt{|a|}\right|\leq\sqrt{|b-a|}$, the desired result follows.

(ii) Observe that
$$\big\langle\varepsilon_{l/n},\delta_{k/n}\big\rangle_{\mathfrak{H}}=\frac{1}{2\sqrt{n}}\left(\sqrt{k+1}-\sqrt{k}-\sqrt{|k+1-l|}+\sqrt{|k-l|}\right).$$
Consequently, for any fixed $l\in\{0,\ldots,n-1\}$, we have
$$\begin{aligned}\sum_{k=0}^{n-1}\Big|\big\langle\varepsilon_{l/n},\delta_{k/n}\big\rangle_{\mathfrak{H}}\Big| &\leq \frac{1}{2}+\frac{1}{2\sqrt{n}}\left(\sum_{k=0}^{l-1}\sqrt{l-k}-\sqrt{l-k-1}\right.\\ &\qquad\left.+1+\sum_{k=l+1}^{n-1}\sqrt{k-l+1}-\sqrt{k-l}\right)\\ &= \frac{1}{2}+\frac{1}{2\sqrt{n}}\big(\sqrt{l}+\sqrt{n-l}\big)\end{aligned}$$
from which we deduce that $\displaystyle\sup_{0\leqslant l\leqslant n-1}\sum_{k=0}^{n-1}\Big|\big\langle\varepsilon_{l/n},\delta_{k/n}\big\rangle_{\mathfrak{H}}\Big| \underset{n\to\infty}{=} O(1)$. It follows that
$$\sum_{k,l=0}^{n-1}\Big|\big\langle\varepsilon_{l/n},\delta_{k/n}\big\rangle_{\mathfrak{H}}\Big|\leqslant n\sup_{0\leqslant l\leqslant n-1}\sum_{k=0}^{n-1}\Big|\big\langle\varepsilon_{l/n},\delta_{k/n}\big\rangle_{\mathfrak{H}}\Big| \underset{n\to\infty}{=} O(n).$$

(iii) We have, by noting $\rho(x) = \frac{1}{2}\big(\sqrt{|x+1|} + \sqrt{|x-1|} - 2\sqrt{|x|}\big)$:

$$\sum_{k,l=0}^{n-1} \left|\langle \delta_{l/n}, \delta_{k/n}\rangle_{\mathfrak{H}}\right|^r \;=\; n^{-r/2} \sum_{k,l=0}^{n-1} |\rho^r(l-k)| \leqslant n^{1-r/2} \sum_{k\in\mathbb{Z}} |\rho^r(k)| \,.$$

Since $\sum_{k\in\mathbb{Z}} |\rho^r(k)| < \infty$ if $r \geqslant 1$, the desired conclusion follows.

(iv) is a consequence of the following identity combined with a telescopic sum argument:

$$\left|\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathfrak{H}} + \frac{1}{2\sqrt{n}}\right| = \frac{1}{2\sqrt{n}}\big(\sqrt{k+1} - \sqrt{k}\big).$$

(v) We have

$$\left|\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathfrak{H}}^2 - \frac{1}{4n}\right| = \frac{1}{4n}\left(\sqrt{k+1} - \sqrt{k}\right)\left|\sqrt{k+1} - \sqrt{k} - 2\right|.$$

Thus, the desired bound is immediately checked by combining a telescoping sum argument with the fact that

$$\left|\sqrt{k+1} - \sqrt{k} - 2\right| = \left|\frac{1}{\sqrt{k+1} + \sqrt{k}} - 2\right| \leqslant 2.$$

$\square$

Also the following lemma will be useful in the sequel:

Lemma 3.2 *Let $\alpha \geqslant 0$ and $q \geqslant 2$ be two positive integers, $g : \mathbb{R}^2 \to \mathbb{R}$ be any function belonging to $\mathscr{C}^{2q}$ and verifying $(\mathbf{H_{2q}})$ defined by (3.15), and $B = (B^{(1)}, B^{(2)})$ be a 2D fractional Brownian motion of Hurst index $1/4$. Set*

$$V_n = n^{-q/4} \sum_{k=0}^{n-1} g(B^{(1)}_{k/n}, B^{(2)}_{k/n})\big(\Delta B^{(1)}_{k/n}\big)^\alpha\, H_q\big(n^{1/4}\Delta B^{(2)}_{k/n}\big),$$

where H_q denotes the qth Hermite polynomial defined by $H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q}\big(e^{-x^2/2}\big)$. Then, the following bound is in order:

$$E\left(|V_n|^2\right) = O(n^{1-q/2-\alpha/2}) \quad \text{as } n \to \infty. \tag{3.16}$$

Proof of Lemma 3.2. We can write

$$
\begin{aligned}
E\left(|V_n|^2\right) &= n^{-q/2}\sum_{k,l=0}^{n-1} E\big[g(B^{(1)}_{k/n},B^{(2)}_{k/n})g(B^{(1)}_{l/n},B^{(2)}_{l/n})\big(\Delta B^{(1)}_{k/n}\big)^{\alpha}\big(\Delta B^{(1)}_{l/n}\big)^{\alpha} \\
&\qquad\qquad\qquad\qquad \times H_q\big(n^{1/4}\Delta B^{(2)}_{k/n}\big)H_q\big(n^{1/4}\Delta B^{(2)}_{l/n}\big)\big] \\
&\underset{(2.11)}{=} \sum_{k,l=0}^{n-1} E\big[g(B^{(1)}_{k/n},B^{(2)}_{k/n})g(B^{(1)}_{l/n},B^{(2)}_{l/n})\big(\Delta B^{(1)}_{k/n}\big)^{\alpha}\big(\Delta B^{(1)}_{l/n}\big)^{\alpha} I^{(2)}_q(\delta^{\otimes q}_{k/n})I^{(2)}_q(\delta^{\otimes q}_{l/n})\big] \\
&\underset{(2.14)}{=} \sum_{r=0}^{q} r!\binom{q}{r}^2 \sum_{k,l=0}^{n-1} E\big[g(B^{(1)}_{k/n},B^{(2)}_{k/n})g(B^{(1)}_{l/n},B^{(2)}_{l/n}) \\
&\qquad \times\big(\Delta B^{(1)}_{k/n}\big)^{\alpha}\big(\Delta B^{(1)}_{l/n}\big)^{\alpha} I^{(2)}_{2q-2r}(\delta^{\otimes q-r}_{k/n}\otimes\delta^{\otimes q-r}_{l/n})\big]\langle\delta_{k/n},\delta_{l/n}\rangle^r_{\mathfrak{H}} \\
&\underset{(2.13)}{=} \sum_{r=0}^{q} r!\binom{q}{r}^2 \sum_{k,l=0}^{n-1} E\Big\langle D^{2q-2r}_{B^{(2)}}\Big(g(B^{(1)}_{k/n},B^{(2)}_{k/n})g(B^{(1)}_{l/n},B^{(2)}_{l/n}) \\
&\qquad \times\ \big(\Delta B^{(1)}_{k/n}\big)^{\alpha}\big(\Delta B^{(1)}_{l/n}\big)^{\alpha}\Big),\delta^{\otimes q-r}_{k/n}\otimes\delta^{\otimes q-r}_{l/n}\Big\rangle_{\mathfrak{H}^{\otimes 2q-2r}}\langle\delta_{k/n},\delta_{l/n}\rangle^r_{\mathfrak{H}} \\
&\underset{(2.10)}{=} \sum_{r=0}^{q} r!\binom{q}{r}^2 \sum_{a+b=2q-2r}\frac{(a+b)!}{a!b!}\sum_{k,l=0}^{n-1} E\left(\frac{d^a g}{dy^a}(B^{(1)}_{k/n},B^{(2)}_{k/n})\frac{d^b g}{dy^b}(B^{(1)}_{l/n},B^{(2)}_{l/n})\right. \\
&\qquad \left.\times\ \big(\Delta B^{(1)}_{k/n}\big)^{\alpha}\big(\Delta B^{(1)}_{l/n}\big)^{\alpha}\right)(2q-2r)!\Big\langle\varepsilon^{\otimes a}_{k/n}\widetilde{\otimes}\varepsilon^{\otimes b}_{l/n},\delta^{\otimes q-r}_{k/n}\otimes\delta^{\otimes q-r}_{l/n}\Big\rangle_{\mathfrak{H}^{\otimes 2q-2r}}\langle\delta_{k/n},\delta_{l/n}\rangle^r_{\mathfrak{H}}.
\end{aligned}
\tag{3.17}
$$

Now, observe that, uniformly in $k,l\in\{0,\ldots,n-1\}$:

$$
\Big\langle\varepsilon^{\otimes a}_{k/n}\widetilde{\otimes}\varepsilon^{\otimes b}_{l/n},\delta^{\otimes q-r}_{k/n}\otimes\delta^{\otimes q-r}_{l/n}\Big\rangle_{\mathfrak{H}^{\otimes 2q-2r}} \underset{n\to\infty}{=} O(n^{-(q-r)}),\ \text{ see Lemma 3.1 (i)},
$$

$$
\left|E\left(\frac{d^a g}{dy^a}(B^{(1)}_{k/n},B^{(2)}_{k/n})\frac{d^b g}{dy^b}(B^{(1)}_{l/n},B^{(2)}_{l/n})\big(\Delta B^{(1)}_{k/n}\big)^{\alpha}\big(\Delta B^{(1)}_{l/n}\big)^{\alpha}\right)\right| \underset{n\to\infty}{=} O(n^{-\alpha/2}),\ \text{use } (\mathbf{H_{2q}}),
$$

and, also:

$$
\sum_{k,l=0}^{n-1}\langle\delta_{k/n},\delta_{l/n}\rangle^r_{\mathfrak{H}} = O(n^{1-r/2})\quad\text{for any fixed } r\geqslant 1,\text{ see Lemma 3.1 (iii).}
$$

Finally, the desired conclusion is obtained by plugging these three bounds into (3.17), after having separated the cases $r=0$ and $r=1$. □

The independent Brownian motion appearing in (1.3) comes from the following proposition.

Proposition 3.3 *Let $(\beta,\widetilde{\beta})$ be a 2D fractional Brownian motion of Hurst index $1/4$. Consider two functions $g,\widetilde{g}:\mathbb{R}^2\to\mathbb{R}$ belonging in $\mathscr{C}^4$, and assume that they both verify $(\mathbf{H_4})$ defined by (3.15). Then*

$$
\begin{aligned}
(G_n,\widetilde{G}_n) &:= \left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} g(\beta_{k/n},\widetilde{\beta}_{k/n})\big(\sqrt{n}(\Delta\beta_{k/n})^2-1\big),\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\widetilde{g}(\beta_{k/n},\widetilde{\beta}_{k/n})\big(\sqrt{n}(\Delta\widetilde{\beta}_{k/n})^2-1\big)\right) \\
&\xrightarrow[n\to\infty]{\text{stably}} \left(\sigma_{1/4}\int_0^1 g(\beta_s,\widetilde{\beta}_s)dW_s+\frac14\int_0^1\frac{\partial^2 g}{\partial x^2}(\beta_s,\widetilde{\beta}_s)ds,\sigma_{1/4}\int_0^1\widetilde{g}(\beta_s,\widetilde{\beta}_s)d\widetilde{W}_s+\frac14\int_0^1\frac{\partial^2\widetilde{g}}{\partial y^2}(\beta_s,\widetilde{\beta}_s)ds\right),
\end{aligned}
$$

where $(W,\widetilde{W})$ is a 2D standard Brownian motion independent of $(\beta,\widetilde{\beta})$, and $\sigma_{1/4}$ is defined by (1.5).

In the particular case where $g(x,y)=g(x)$ and $\widetilde{g}(x,y)=\widetilde{g}(y)$, the conclusion of the proposition follows directly from (1.8). In the general case, the proof only consists to extend literaly the proof of (1.8) contained in [16]. Details are left to the reader.

4 Proof of Theorem 1.2

We are now in position to prove our main result, that is Theorem 1.2.

Proof of the third point (case $H<1/4$). Firstly, observe that (1.4) is actually a particular case of the following result, which is valid for any fractional Brownian β with Hurst index H belonging to $(0,3/4)$:

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor n\cdot\rfloor-1}\left(n^{2H}(\Delta\beta_{k/n})^2-1\right)\overset{\text{stably}}{\underset{n\to\infty}{\longrightarrow}}\sigma_H\,W$$

with W an independent Brownian motion and $\sigma_H>0$ an (explicit) constant. By mimicking the proof contained in the fourth point of Remark 1.3, we get, here, for any $H\in(0,3/4)$,

$$n^{2H-1/2}\sum_{k=0}^{\lfloor n\cdot\rfloor-1}\Delta B^{(1)}_{k/n}\Delta B^{(2)}_{k/n}\overset{\text{stably}}{\underset{n\to\infty}{\longrightarrow}}\frac{\sigma_H}{\sqrt{2}}W. \tag{4.18}$$

But, see (1.7), the existence of $\int_0^\cdot B_s\cdot d^\star B_s$ would imply in particular that $\sum_{k=0}^{\lfloor n\cdot\rfloor-1}\Delta B^{(1)}_{k/n}\Delta B^{(2)}_{k/n}$ converges in law as $n\to\infty$, which is in contradiction with (4.18) for $H<1/4$. The proof of the third point is done.

Proof of the second point (case $H=1/4$). For the simplicity of the exposition, we assume from now that $t=1$, the general case being of course similar up to cumbersome notation. For any $a,b,c,d\in\mathbb{R}$, by the classical Taylor formula, we can expand $f(b,d)$ as (compare with (1.7)):

$$\begin{aligned}
& f(a,c)+\partial_1 f(a,c)(b-a)+\partial_2 f(a,c)(d-c)+\frac{1}{2}\partial_{11}f(a,c)(b-a)^2+\frac{1}{2}\partial_{22}f(a,c)(d-c)^2\\
+\;& \frac{1}{6}\partial_{111}f(a,c)(b-a)^3+\frac{1}{6}\partial_{222}f(a,c)(d-c)^3+\frac{1}{24}\partial_{1111}f(a,c)(b-a)^4+\frac{1}{24}\partial_{2222}f(a,c)(d-c)^4\\
+\;& \partial_{12}f(a,c)(b-a)(d-c)+\frac{1}{2}\partial_{112}f(a,c)(b-a)^2(d-c)+\frac{1}{2}\partial_{122}f(a,c)(b-a)(d-c)^2\\
+\;& \frac{1}{6}\partial_{1112}f(a,c)(b-a)^3(d-c)+\frac{1}{4}\partial_{1122}f(a,c)(b-a)^2(d-c)^2+\frac{1}{6}\partial_{1222}f(a,c)(b-a)(d-c)^3
\end{aligned} \tag{4.19}$$

plus a remainder term. Here, as usual, the notation $\partial_{1\ldots12\ldots2}f$ (where the index 1 is repeated k times and the index 2 is repeated l times) means that f is differentiated k times w.r.t. the first component and l times w.r.t. the second one. By combining (4.19) with the following identity, available for any $h:\mathbb{R}\to\mathbb{R}$ belonging to $\mathscr{C}^4$:

$$\begin{aligned} & h'(a)(b-a)+\frac{1}{2}h''(a)(b-a)^2+\frac{1}{6}h'''(a)(b-a)^3+\frac{1}{24}h''''(a)(b-a)^4 \\ = \; & \frac{h'(a)+h'(b)}{2}(b-a)-\frac{1}{12}h'''(a)(b-a)^3-\frac{1}{24}h''''(a)(b-a)^4+\text{ some remainder} \end{aligned}$$

we get that $f(b,d)$ can also be expanded as

$$\begin{aligned} & f(a,c)+\frac{1}{2}\big(\partial_1 f(a,c)+\partial_1 f(b,c)\big)(b-a)-\frac{1}{12}\partial_{111}f(a,c)(b-a)^3-\frac{1}{24}\partial_{1111}f(a,c)(b-a)^4 \\ + \; & \frac{1}{2}\big(\partial_2 f(a,c)+\partial_2 f(a,d)\big)(d-c)-\frac{1}{12}\partial_{222}f(a,c)(d-c)^3-\frac{1}{24}\partial_{2222}f(a,c)(d-c)^4 \\ + \; & \partial_{12}f(a,c)(b-a)(d-c)+\frac{1}{2}\partial_{112}f(a,c)(b-a)^2(d-c)+\frac{1}{2}\partial_{122}f(a,c)(b-a)(d-c)^2 \\ + \; & \frac{1}{6}\partial_{1112}f(a,c)(b-a)^3(d-c)+\frac{1}{4}\partial_{1122}f(a,c)(b-a)^2(d-c)^2+\frac{1}{6}\partial_{1222}f(a,c)(b-a)(d-c)^3 \end{aligned} \tag{4.20}$$

plus a remainder term.

By setting $a=B^{(1)}_{k/n}$, $b=B^{(1)}_{(k+1)/n}$, $c=B^{(2)}_{k/n}$ and $d=B^{(2)}_{(k+1)/n}$ in (4.20), and by suming the obtained expression for k over $0,\dots,n-1$, we deduce that the conclusion in Theorem 1.2 is a consequence of the following convergences:

$$S_n^{(1)} := \sum_{k=0}^{n-1}\partial_{111}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\big(\Delta B^{(1)}_{k/n}\big)^3 \xrightarrow[n\to\infty]{L^2} -\frac{3}{2}\int_0^1 \partial_{1111}f(B^{(1)}_s,B^{(2)}_s)ds \tag{4.21}$$

$$S_n^{(2)} := \sum_{k=0}^{n-1}\partial_{1111}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\big(\Delta B^{(1)}_{k/n}\big)^4 \xrightarrow[n\to\infty]{L^2} 3\int_0^1 \partial_{1111}f(B^{(1)}_s,B^{(2)}_s)ds \tag{4.22}$$

$$S_n^{(3)} := \sum_{k=0}^{n-1}\partial_{222}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\big(\Delta B^{(2)}_{k/n}\big)^3 \xrightarrow[n\to\infty]{L^2} -\frac{3}{2}\int_0^1 \partial_{2222}f(B^{(1)}_s,B^{(2)}_s)ds \tag{4.23}$$

$$S_n^{(4)} := \sum_{k=0}^{n-1}\partial_{2222}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\big(\Delta B^{(2)}_{k/n}\big)^4 \xrightarrow[n\to\infty]{L^2} 3\int_0^1 \partial_{2222}f(B^{(1)}_s,B^{(2)}_s)ds \tag{4.24}$$

$$\begin{aligned} S_n^{(5)} := \sum_{k=0}^{n-1}\partial_{12}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\Delta B^{(1)}_{k/n}\,\Delta B^{(2)}_{k/n} \xrightarrow[n\to\infty]{\text{stably}} & \frac{\sigma_{1/4}}{\sqrt{2}}\int_0^1 \partial_{12}f(B^{(1)}_s,B^{(2)}_s)dW_s \\ & +\frac{1}{4}\int_0^1 \partial_{1122}f(B^{(1)}_s,B^{(2)}_s)ds \end{aligned} \tag{4.25}$$

$$S_n^{(6)} := \sum_{k=0}^{n-1}\partial_{112}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\big(\Delta B^{(1)}_{k/n}\big)^2\,\Delta B^{(2)}_{k/n} \xrightarrow[n\to\infty]{L^2} -\frac{1}{2}\int_0^1 \partial_{1122}f(B^{(1)}_s,B^{(2)}_s)ds \tag{4.26}$$

$$S_n^{(7)} := \sum_{k=0}^{n-1}\partial_{122}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\Delta B^{(1)}_{k/n}\,\big(\Delta B^{(2)}_{k/n}\big)^2 \xrightarrow[n\to\infty]{L^2} -\frac{1}{2}\int_0^1 \partial_{1122}f(B^{(1)}_s,B^{(2)}_s)ds \tag{4.27}$$

$$S_n^{(8)} := \sum_{k=0}^{n-1} \partial_{1122} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \big(\Delta B_{k/n}^{(1)}\big)^2 \big(\Delta B_{k/n}^{(2)}\big)^2 \xrightarrow[n\to\infty]{L^2} \int_0^1 \partial_{1122} f(B_s^{(1)}, B_s^{(2)}) ds \quad (4.28)$$

$$S_n^{(9)} := \sum_{k=0}^{n-1} \partial_{1112} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \big(\Delta B_{k/n}^{(1)}\big)^3 \Delta B_{k/n}^{(2)} \xrightarrow[n\to\infty]{\text{Prob}} 0 \quad (4.29)$$

$$S_n^{(10)} := \sum_{k=0}^{n-1} \partial_{1222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(1)} \big(\Delta B_{k/n}^{(2)}\big)^3 \xrightarrow[n\to\infty]{\text{Prob}} 0. \quad (4.30)$$

Note that the term corresponding to the remainder in (4.20) converges in probability to zero due to the fact that B has a finite quartic variation.

Proof of (4.21), (4.23), (4.26) and (4.27). By Lemma 3.2 with $q = 3$ and $\alpha = 0$, and by using the basic fact that

$$\big(\Delta B_{k/n}^{(2)}\big)^3 = n^{-3/4} H_3(n^{1/4} \Delta B_{k/n}^{(2)}) + \frac{3}{\sqrt{n}} \Delta B_{k/n}^{(2)}, \quad (4.31)$$

we immediately see that (4.23) is a consequence of the following convergence:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(2)} \xrightarrow[n\to\infty]{L^2} -\frac{1}{2} \int_0^1 \partial_{2222} f(B_s^{(1)}, B_s^{(2)}) ds. \quad (4.32)$$

So, let us prove (4.32). We have, on one hand:

$$\begin{aligned}
& E\left| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)}) \Delta B_{k/n}^{(2)} \right|^2 \\
= \;& \frac{1}{n} \sum_{k,l=0}^{n-1} E\Big(\partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)})\, \partial_{222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)})\, \Delta B_{k/n}^{(2)}\, \Delta B_{l/n}^{(2)} \Big) \\
= \;& \frac{1}{n} \sum_{k,l=0}^{n-1} E\Big(\partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)})\, \partial_{222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)})\, I_2^{(2)}(\delta_{k/n} \otimes \delta_{l/n}) \Big) \\
+ \;& \frac{1}{n} \sum_{k,l=0}^{n-1} E\Big(\partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)})\, \partial_{222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \Big) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \\
= \;& \frac{1}{n} \sum_{k,l=0}^{n-1} E\Big(\partial_{22222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)})\, \partial_{222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \Big) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \\
+ \;& \frac{1}{n} \sum_{k,l=0}^{n-1} E\Big(\partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)})\, \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \Big) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{l/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \\
+ \;& \frac{1}{n} \sum_{k,l=0}^{n-1} E\Big(\partial_{2222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)})\, \partial_{2222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \Big) \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \\
+ \;& \frac{1}{n} \sum_{k,l=0}^{n-1} E\Big(\partial_{222} f(B_{k/n}^{(1)}, B_{k/n}^{(2)})\, \partial_{22222} f(B_{l/n}^{(1)}, B_{l/n}^{(2)}) \Big) \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{l/n}, \delta_{l/n} \rangle_{\mathfrak{H}}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{n}\sum_{k,l=0}^{n-1} E\left(\partial_{222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\, \partial_{222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\right) \langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathfrak{H}} \\
&= \quad a^{(n)} + b^{(n)} + c^{(n)} + d^{(n)} + e^{(n)}.
\end{aligned}
$$

Using Lemma 3.1 (i) and (ii), we have that $a^{(n)}$, $c^{(n)}$ and $d^{(n)}$ tends to zero as $n \to \infty$. Using Lemma 3.1 (iii), we have that $e^{(n)}$ tends to zero as $n \to \infty$. Finally, observe that

$$
\begin{aligned}
b^{(n)} \quad &= \quad \frac{1}{4n^2}\sum_{k,l=0}^{n-1} E\left(\partial_{2222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\, \partial_{2222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\right) \\
&- \quad \frac{1}{2n\sqrt{n}}\sum_{k,l=0}^{n-1} E\left(\partial_{2222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\, \partial_{2222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\right)\left(\langle \varepsilon_{l/n}, \delta_{l/n}\rangle_{\mathfrak{H}} + \frac{1}{2\sqrt{n}}\right) \\
&+ \quad \frac{1}{n}\sum_{k,l=0}^{n-1} E\left(\partial_{2222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\, \partial_{2222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\right)\left(\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathfrak{H}} + \frac{1}{2\sqrt{n}}\right)\langle \varepsilon_{l/n}, \delta_{l/n}\rangle_{\mathfrak{H}}.
\end{aligned}
$$

Therefore, using Lemma 3.1 (i) and (iv), we have

$$
E\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} \partial_{222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\Delta B^{(2)}_{k/n}\right|^2 = E\left|\frac{1}{2n}\sum_{k=0}^{n-1} \partial_{2222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\right|^2 + o(1). \tag{4.33}
$$

On the other hand, we have

$$
\begin{aligned}
& E\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} \partial_{222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\Delta B^{(2)}_{k/n} \times \frac{-1}{2n}\sum_{l=0}^{n-1} \partial_{2222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\right) \\
&= \quad -\frac{1}{2n\sqrt{n}}\sum_{k,l=0}^{n-1} E\big(\partial_{222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\partial_{2222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\, \Delta B^{(2)}_{k/n}\big) \\
&= \quad -\frac{1}{2n\sqrt{n}}\sum_{k,l=0}^{n-1} E\big(\partial_{2222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\partial_{2222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\big)\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathfrak{H}} \\
&- \quad \frac{1}{2n\sqrt{n}}\sum_{k,l=0}^{n-1} E\big(\partial_{222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\partial_{22222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\big)\langle \varepsilon_{l/n}, \delta_{k/n}\rangle_{\mathfrak{H}} \\
&= \quad \frac{1}{4n^2}\sum_{k,l=0}^{n-1} E\big(\partial_{2222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\partial_{2222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\big) \\
&- \quad \frac{1}{2n\sqrt{n}}\sum_{k,l=0}^{n-1} E\big(\partial_{2222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\partial_{2222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\big)\left(\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathfrak{H}} + \frac{1}{2\sqrt{n}}\right) \\
&- \quad \frac{1}{2n\sqrt{n}}\sum_{k,l=0}^{n-1} E\big(\partial_{222} f(B^{(1)}_{k/n}, B^{(2)}_{k/n})\partial_{22222} f(B^{(1)}_{l/n}, B^{(2)}_{l/n})\big)\langle \varepsilon_{l/n}, \delta_{k/n}\rangle_{\mathfrak{H}}
\end{aligned}
$$

We immediately have that the second (see Lemma 3.1 (iv)) and the third (see Lemma 3.1 (ii)) terms in the previous expression tends to zero as $n \to \infty$. That is

$$\begin{aligned} & E\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\partial_{222}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\Delta B^{(2)}_{k/n}\times\frac{-1}{2n}\sum_{l=0}^{n-1}\partial_{2222}f(B^{(1)}_{l/n},B^{(2)}_{l/n})\right) \\ = \; & E\left|\frac{1}{2n}\sum_{k=0}^{n-1}\partial_{2222}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\right|^2+o(1). \end{aligned} \tag{4.34}$$

We have proved, see (4.33) and (4.34), that

$$E\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\partial_{222}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\Delta B^{(2)}_{k/n}+\frac{1}{2n}\sum_{k=0}^{n-1}\partial_{2222}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\right|^2\underset{n\to\infty}{\longrightarrow}0.$$

This implies (4.32).

The proof of (4.21) follows directly from (4.23) by exchanging the roles played by $B^{(1)}$ and $B^{(2)}$. On the other hand, by combining Lemma 3.2 with the following basic identity:

$$\big(\Delta B^{(2)}_{k/n}\big)^2=\frac{1}{\sqrt{n}}\,H_2(n^{1/4}\Delta B^{(2)}_{k/n})+\frac{1}{\sqrt{n}},$$

we see that (4.27) is also a direct consequence of (4.32). Finally, (4.26) is obtained from (4.27) by exchanging the roles played by $B^{(1)}$ and $B^{(2)}$.

Proof of (4.22), (4.24) and (4.28). By combining Lemma 3.2 with the identity

$$\big(\Delta B^{(1)}_{k/n}\big)^4=\frac{1}{n}\,H_4(n^{1/4}\Delta B^{(1)}_{k/n})+\frac{6}{n}\,H_2(n^{1/4}\Delta B^{(1)}_{k/n})+\frac{3}{n},$$

we see that (4.24) is easily obtained through a Riemann sum argument. We can use the same arguments in order to prove (4.22). Finally, to obtain (4.28), it suffices to combine Lemma 3.2 with the identity

$$\big(\Delta B^{(1)}_{k/n}\big)^2\big(\Delta B^{(2)}_{k/n}\big)^2=\frac{1}{n}+\frac{1}{\sqrt{n}}\,\big(\Delta B^{(1)}_{k/n}\big)^2\,H_2(n^{1/4}\Delta B^{(2)}_{k/n})+\frac{1}{n}\,H_2(n^{1/4}\Delta B^{(1)}_{k/n}).$$

Proof of (4.29) and (4.30). We only prove (4.30), the proof of (4.29) being obtained from (4.30) by exchanging the roles played by $B^{(1)}$ and $B^{(2)}$. By combining (4.31) with Lemma 3.2, it suffices to prove that

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\partial_{1222}f(B^{(1)}_{k/n},B^{(2)}_{k/n})\,\Delta B^{(1)}_{k/n}\,\Delta B^{(2)}_{k/n}\overset{\text{Prob}}{\underset{n\to\infty}{\longrightarrow}}0.$$

But this last convergence follows directly from Lemma 3.3. Therefore, the proof of (4.30) is done.

Proof of (4.25). We combine Proposition 3.3 with the idea developed in the third comment that we have addressed just after the statement of Theorem 1.2. Indeed, we have

$$\begin{aligned}
&\sum_{k=0}^{n-1} \partial_{12} f\big(B^{(1)}_{k/n}, B^{(2)}_{k/n}\big)\Delta B^{(1)}_{k/n}\Delta B^{(2)}_{k/n} \\
=\;& \frac{1}{2\sqrt{n}}\sum_{k=0}^{n-1} \partial_{12} f\Big(\frac{\beta_{k/n}+\widetilde{\beta}_{k/n}}{\sqrt{2}}, \frac{\beta_{k/n}-\widetilde{\beta}_{k/n}}{\sqrt{2}}\Big)\Big(\sqrt{n}\big(\Delta\beta_{k/n}\big)^2-1\Big) \\
&-\frac{1}{2\sqrt{n}}\sum_{k=0}^{n-1} \partial_{12} f\Big(\frac{\beta_{k/n}+\widetilde{\beta}_{k/n}}{\sqrt{2}}, \frac{\beta_{k/n}-\widetilde{\beta}_{k/n}}{\sqrt{2}}\Big)\Big(\sqrt{n}\big(\Delta\widetilde{\beta}_{k/n}\big)^2-1\Big)
\end{aligned}$$

for $\beta = (B^{(1)} + B^{(2)})/\sqrt{2}$ and $\widetilde{\beta} = (B^{(1)} - B^{(2)})/\sqrt{2}$. Note that $(\beta, \widetilde{\beta})$ is also a 2D fractional Brownian motion of Hurst index 1/4. Hence, using Proposition 3.3 with $g(x, y) = \widetilde{g}(x, y) = f\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right)$, we get

$$\begin{aligned}
&\sum_{k=0}^{n-1} \partial_{12} f\big(B^{(1)}_{k/n}, B^{(2)}_{k/n}\big)\Delta B^{(1)}_{k/n}\Delta B^{(2)}_{k/n} \\
\underset{n\to\infty}{\overset{\text{stably}}{\longrightarrow}}\;& \frac{\sigma_{1/4}}{2}\int_0^1 \partial_{12} f(B^{(1)}_s, B^{(2)}_s)d(W-\widetilde{W})_s + \frac{1}{4}\int_0^1 \partial_{1122} f(B^{(1)}_s, B^{(2)}_s)ds \\
\overset{\text{Law}}{=}\;& \frac{\sigma_{1/4}}{\sqrt{2}}\int_0^1 \partial_{12} f(B^{(1)}_s, B^{(2)}_s)dW_s + \frac{1}{4}\int_0^1 \partial_{1122} f(B^{(1)}_s, B^{(2)}_s)ds,
\end{aligned}$$

for $(W, \widetilde{W})$ a 2D standard Brownian motion independent of $(\beta, \widetilde{\beta})$. The proof of (4.25) is done.

Proof of the first point (case $H > 1/4$). The proof can be done by following exactly the same strategy than in the step above. The only difference is that, using a version of Lemma 3.2 together with computations similar to that allowing to obtain (4.32), the limits in (4.21)–(4.28) are, here, *all equal to zero* (for the sake of simplicity, the technical details are left to the reader). Therefore, we can deduce (1.2) by using (4.20). □

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